PARTIAL REDUCTIONS OF HAMILTONIAN FLOWS AND HESS-APPEL'ROT SYSTEMS ON $SO(n)$

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Abstract. We study reductions of the Hamiltonian flows restricted to their invariant submanifolds. As examples, we consider partial Lagrange-Routh reductions of the natural mechanical systems such as geodesic flows on compact Lie groups and n-dimensional variants of the classical Hess-Appel'rot case of a heavy rigid body motion about a fixed point.
MSC: 37J35, 37J15, 53D20

Contents

1. Introduction

In this paper we study reductions of the Hamiltonian flows restricted to their invariant submanifolds. Apparently, the lowering of order in Hamiltonian systems having invariant relations was firstly studied by Levi-Civita (e.g., see [17], ch. X).

1.1. **Hess-Appel'rot System.** The classical example of the system having an invariant relation is a celebrated Hess-Appel'rot case of a heavy rigid body motion

[15, 1]. Recall that the motion of a heavy rigid body around a fixed point, in the moving frame, is represented by the Euler-Poisson equations

$$\frac{d}{dt}\vec{m} = \vec{m} \times \vec{\omega} + \mathfrak{G}\mathfrak{M}\,\vec{\gamma} \times \vec{r}, \quad \frac{d}{dt}\vec{\gamma} = \vec{\gamma} \times \vec{\omega}, \quad \vec{\omega} = \mathrm{A}\vec{m}, \tag{1}$$

where $\vec{\omega}$ is the angular velocity, $\vec{m}$ the angular momentum, $\mathrm{I} = \mathrm{A}^{-1}$ the inertial tensor, $\mathfrak{M}$ mass and $\vec{r}$ the vector of the mass center of a rigid body; $\vec{\gamma}$ is the direction of the homogeneous gravitational field and $\mathfrak{G}$ is the gravitational constant.

The equations (1) always have three integrals, the energy, geometric integral and the projection of angular momentum:

$$\mathcal{F}_1 = \frac{1}{2}(\vec{m}, \vec{\omega}) + \mathfrak{M}\mathfrak{G}(\vec{r}, \vec{\gamma}), \quad \mathcal{F}_2 = (\vec{\gamma}, \vec{\gamma}) = 1, \quad \mathcal{F}_3 = (\vec{m}, \vec{\gamma}). \tag{2}$$

For the integrability we need a forth integral. There are three famous integrable cases: Euler, Lagrange and Kowalevskaya [2, 14].

Apart of these cases, there are various particular solutions (e.g., see [14]). The celebrated is partially integrable Hess-Appel'rot case [15, 1]. Under the conditions:

$$r_2 = 0, \quad r_1\sqrt{a_3 - a_2} \pm r_3\sqrt{a_2 - a_1} = 0, \quad \mathrm{A} = \mathrm{diag}(a_1, a_2, a_3), \tag{3}$$

$a_3 > a_2 > a_1 > 0$, system (1) has an invariant relation given by

$$\mathcal{F}_4 = (\vec{m}, \vec{r}) = r_1 m_1 + r_3 m_3 = 0. \tag{4}$$

The system is integrable up to one quadrature: the compact connected components of the regular invariant sets $\mathcal{F}_1 = c_1, \mathcal{F}_2 = 1, \mathcal{F}_3 = c_3, \mathcal{F}_4 = 0$ are tori, but not with quasi-periodic dynamics. The classical and algebro-geometric integration can be found in [14] and [9], respectively.

There is a nice geometrical interpretation of the conditions (3): the intersection of the plane orthogonal to $\vec{r}$ with the ellipsoid $(\mathrm{A}\vec{m}, \vec{m}) = const$ is a circle (e.g., see [8]). If instead of the moving base given by the main axes of the inertia, we take the moving base $\vec{f}_1, \vec{f}_2, \vec{f}_3$ such that the mass center of a rigid body $\vec{r}$ is proportional to $\vec{f}_3$,

$$\vec{r} = \rho \vec{f}_3, \quad \rho = \sqrt{r_1^2 + r_3^2}, \tag{5}$$

then the inverse of the inertial operator reads

$$\mathrm{A} = \begin{pmatrix} a_2 & 0 & a_{13} \\ 0 & a_2 & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \tag{6}$$

and the invariant relation (4) is simply given by

$$m_3 = 0. \tag{7}$$

Historical overview, with an application of Levi-Civita ideas to the Hess-Appel'rot system and other classical rigid body problems can be found in Borisov and Mamaev [8].

1.2. **Reduction of Symmetries.** Let G be a Lie group with a free proper Hamiltonian action on a symplectic manifold (M,ω). Let

$$\Phi : M \to \mathfrak{g}^* \tag{8}$$

be the corresponding equivariant momentum map. Assume that η is a regular value of Φ, so that

$$M_\eta = \Phi^{-1}(\eta) \tag{9}$$

and $M_{\mathcal{O}_\eta} = \Phi^{-1}(\mathcal{O}_\eta)$ are smooth manifolds. Here $\mathcal{O}_\eta = G/G_\eta$ is the coadjoint orbit of η. The manifolds M_η and $M_{\mathcal{O}_\eta}$ are G_η-invariant and G-invariant, respectively.

There is a unique symplectic structure ω_η on $N_\eta = M_\eta/G_\eta$ satisfying

$$\omega|_{M_\eta} = d\pi_\eta^*\omega_\eta \tag{10}$$

where $\pi_\eta : M_\eta \to N_\eta$ is the natural projection [20].

According to Noether's theorem, if h is a G-invariant function then the momentum mapping Φ is an integral of the Hamiltonian system

$$\dot{x} = X_h. \tag{11}$$

In addition, its restriction to the invariant submanifold M_η projects to the Hamiltonian system $\dot{y} = X_H$ on the reduced space N_η with H defined by [20]

$$h|_{M_\eta} = H \circ \pi_\eta\,. \tag{12}$$

An alternative description of the reduced space is as follows. Let $\{\cdot,\cdot\}$ be the canonical Poisson bracket on (M,ω). Then the manifold M/G carries the induced Poisson structure and $M_{\mathcal{O}_\eta}/G$ is the symplectic leaf in M/G. The mapping

$$\Psi : N_\eta \to M_{\mathcal{O}_\eta}/G \tag{13}$$

which assigns to the G_η-orbit of $x \in M_\eta$ the G-orbit through x in $M_{\mathcal{O}_\eta}$ establish the symplectomorphism between N_η and $M_{\mathcal{O}_\eta}/G$ (e.g., see [24]).

1.3. **Outline and Results of the Paper.** The aim of this paper is to study Hamiltonian systems that naturally generalize geometrical properties of the Hess-Appel'rot system (Definition 3, Section 4). This is the reason we are interested in the following: suppose that h is not a G-invariant function, but (9) is still an invariant manifold of the Hamiltonian system (11). As a modification of the regular Marsden-Weinstein reduction, we study (partial) reduction of the Hamiltonian system (11) from M_η to N_η and relationship between the integrability of the reduced and nonreduced system (Theorems 1, 2, Section 2).

In Section 3 we define partial Lagrange-Routh reductions (Theorem 3). The construction of the geodesic flows on compact Lie groups, having invariant manifolds foliated by invariant tori is presented. However, the flows over invariant tori are not quasi-periodic.

In Section 4 we consider an n-dimensional heavy rigid body motion about a fixed point. Recall that there are two natural multidimensional variants of a heavy rigid body which are described by the Euler-Poisson equations on the dual spaces of semi-direct products $so(n) \times so(n)$ and $so(n) \times \mathbb{R}^n$. The integrable generalizations of the Lagrange top with respect to the first and the second n-dimensional variant are given in [25, 10] and [3], respectively. The n-dimensional Kowalevskaya top in the presence of q gravitational fields ($q \leq n$) represented by Euler-Poisson equations on the dual space of the semi-direct product $so(n) \times (\mathbb{R}^n)^q$ is given in [26].

Recently, an interesting construction of the Hess-Appel'rot system on $so(n) \times so(n)$ is studied by Dragović and Gajić [11]. Besides, by generalizing analytical and algebraic properties of the classical system (1), (3), they introduced a class of systems with invariant relations, so called *Hess-Appel'rot type systems* with remarkable property: there exist a pair of compatible Poisson structures, such that the system is Hamiltonian with respect to the first structure and invariant relations are Casimir functions with respect to the second structure (for more details see [11]).

We consider the another generalization of a heavy rigid body and define the n-dimensional Hess-Appel'rot system within the framework of partial reductions (Lemma 6). It is known that the motion of a rigid body mass center in the classical Hess-Appel'rot system is described by the spherical pendulum equations [29, 8]. We shall prove the same statement for the n-dimensional variant of the system (Theorem 7). On the other side, the L-A pair of the system written in the closed form of the Euler-Poisson equations on the dual space of the semi-direct product $so(n) \times \mathbb{R}^n$ is given. In particular, we get the L-A pair representation for the Euler-Poisson equations of the classical Hess-Appel'rot system (1), (5), (6) different from those given in [9] (Theorem 9).

Finally, we have made a relationship between our construction and the Dragović-Gajić $so(4) \times so(4)$ system (Theorem 10, Section 5).

2. Partial Reductions and Integrability

Let G be a compact connected Lie group with a free Hamiltonian action on a symplectic manifold (M, ω) with the momentum map (8). Assume that η is a regular value of Φ.

Theorem 1. (i) *Suppose that the restriction of h to $M_{\mathcal{O}_\eta}$ is a G-invariant function. Then M_η is an invariant manifold of the Hamiltonian system (11) and $X_h|_{M_\eta}$ projects to the Hamiltonian vector field X_H*

$$d\pi_\eta(X_h)|_x = X_H|_{y=\pi_\eta(x)}, \tag{14}$$

where H is the induced function on N_η defined by (12).

(ii) *The inverse statement also holds: if (9) is an invariant submanifold of the Hamiltonian system (11) then the restriction of h to M_η is a G_η-invariant function and $X_h|_{M_\eta}$ projects to the Hamiltonian vector field X_H on N_η, where H is defined by (12).*

In both cases, the Hamiltonian vector field X_h is not assumed to be G-invariant on M. Moreover $X_h|_{M_\eta}$ may not be G_η-invariant as well. It is invariant modulo the kernel of $d\pi_\eta$, which is sufficient the tools of symplectic reduction are still applicable.

Definition 1. We shall refer to the passing from $\dot{x} = X_h|_{M_\eta}$ to

$$\dot{y} = X_H \tag{15}$$

as a *partial reduction*.

If h is G-invariant, the complete integrability of the reduced and original system are closely related (see [16, 28]). In our case, we have the following corollary.

Theorem 2. (i) *Suppose that the partially reduced system (15) is completely integrable, with a complete set of commuting integrals $F_1, \dots, F_m$, $m = \frac{1}{2}\dim N_\eta$.*

Let $f_1 = F_1 \circ \pi_\eta, \dots, f_n = F_n \circ \pi_\eta$. Then M_η is almost everywhere foliated by $(m + \dim G_\eta)$-dimensional invariant isotropic manifolds

$$\mathcal{M}_c = \{f_1 = c_1, \dots, f_m = c_m\} \tag{16}$$

of the system (11).

(ii) *In addition, if η is a regular element of $\mathfrak{g}^*$ and $f_1, \dots, f_m$ can be extended to the commuting G-invariant functions in some neighborhood of M_η, then the compact connected components of invariant manifolds (16) are tori. However, in general, the flow over the tori is not quasi-periodic.*

Remark 1. If the partially reduced system (15) is integrable then we need $(\dim G_\eta)$-additional quadratures for the solving of $\dot{x} = X_h|_{M_\eta}$ (*the reconstruction equations*). Also, if $G_\eta = G$, i.e., $\mathcal{O}_\eta = \{\eta\}$, then $m + \dim G_\eta = \frac{1}{2}\dim M$. Whence, the invariant manifolds (16) are Lagrangian.

The partial reduction can be seen as a special case of the symplectic reductions studied in [4, 19] (see also [18], Ch. III). There, the *symplectic reduction* of a symplectic manifold (M, ω) is any surjective submersion $p : N \to P$ of a submanifold $N \subset M$ onto another symplectic manifold (P, Ω), which satisfies $p^*\Omega = \omega|_N$. Also, instead of a compact group action one can consider a proper group action. However we work in the framework of a regular Marsden-Weinstein reduction and a compact group action, which allows us to easily describe the partial reductions of natural mechanical systems considered in this paper.

Proof of Theorem 1. (i) Let $\xi_1, \dots, \xi_n$ be the base of $\mathfrak{g}$, $(\eta, \xi_\alpha) = \eta_\alpha$ and

$$\phi_\alpha = (\Phi, \xi_\alpha) : M \to \mathbb{R}, \quad \alpha = 1, \dots, n. \tag{17}$$

Then the level set of the momentum mapping (9) is given by the equations

$$\phi_\alpha = \eta_\alpha, \quad \alpha = 1, \dots, n.$$

The action of G is generated by Hamiltonian vector fields X_{ϕ_α}. Since $h|_{M_{\mathcal{O}_\eta}}$ is G-invariant, for $x \in M_\eta$ we have

$$(dh, X_{\phi_\alpha}) = \{h, \phi_\alpha\} = -(d\phi_\alpha, X_h) = 0, \quad \alpha = 1, \dots, n. \tag{18}$$

Thus M_η is an invariant submanifold.

Let h^* be an arbitrary G-invariant function that coincides with h on $M_{\mathcal{O}_\eta}$. Then X_{h^*} is a G_η-invariant vector field on M_η which project to X_H [20]:

$$d\pi_\eta(X_{h^*})|_x = X_H|_{y=\pi_\eta(x)}. \tag{19}$$

Let $\delta = h - h^*$. From the condition $\delta|_{M_\eta} = 0$ we can express δ, in a an open neighborhood of M_η, as

$$\delta(x) = \sum_{\alpha=1}^{n} \delta_\alpha(x)\left(\phi_\alpha(x) - \eta_\alpha\right).$$

Now, let f be a G-invariant function on M. Since $\{f, \phi_\alpha\} = 0$ (Noether's theorem), we get

$$\{\delta, f\}|_{M_\eta} = \left(\sum_\alpha \{\delta_\alpha, f\}\left(\phi_\alpha - \eta_\alpha\right) + \sum_\alpha \delta_\alpha\{\phi_\alpha, f\}\right)|_{M_\eta} = 0.$$

Thus the Poisson bracket $\{\delta, f\}|_{M_\eta} = -(df, X_\delta)|_{M_\eta}$ vanish for an arbitrary G-invariant function f. In other words

$$X_\delta|_x \in T_x(G\cdot x)\cap T_x M_\eta \,. \tag{20}$$

Combining (19), $X_h = X_{h^*} + X_\delta$ and (20) with the well known identity (e.g, see [24])

$$T_x(G\cdot x)\cap T_x M_\eta = T_x(G_\eta \cdot x) = \ker d\pi_\eta|_x \,,$$

we prove the relation (14).

(ii) If G is a connected compact group, the coadjoint isotropy group G_η is connected. Thus, the function h is G_η-invariant if and only if it is invariant with respect to the infinitesimal action of G_η.

Suppose that M_η is an invariant submanifold of (11). Then (18) holds for $x\in M_\eta$. Since the action of G is generated by Hamiltonian vector fields X_{ϕ_α}, from (18) we get that h is invariant with repsect to the infinitesimal action of G_η. Therefore we have well defined reduced Hamiltonian function H on N_η.

By using the diffeomorphism (13) and the fact that $M_{\mathcal{O}_\eta}$ is a closed submanifold of M, we can find a G-invariant function h^* on M which coincides with h on M_η. Now, the relation (14) follows from the proof of item (i). □

Proof of Theorem 2. (i) Consider a regular invariant Lagrangian submanifold

$$\mathcal{N}_c = \{F_1 = c_1, \dots, F_m = c_m\} \subset N_\eta, \tag{21}$$

of the partially reduced system (15). From the relations (10) and (14) we get that $\mathcal{M}_c = \pi_\eta^{-1}(\mathcal{N}_c)$ is an invariant isotropic manifold of the system (11).

(ii) Let $\eta\in\mathfrak{g}^*$ be a regular element ($G_\eta \approx \mathbb{T}^r$ is a maximal torus of G, $r = \operatorname{rank} G$). If the connected component $\mathcal{N}_c^o$ of (21) is compact then, by Liouville's theorem, it is diffeomorphic to a m-dimensional torus $\mathbb{T}^m$ with quasi-periodic flow of (15). Thus the compact connected component $\mathcal{M}_c^o = \pi_\eta^{-1}(\mathcal{N}_c^o)$ of (16) is a *torus bundle* over $\mathbb{T}^m$:

$$\begin{array}{ccc} \mathbb{T}^r & \longrightarrow & \mathcal{M}_c^o \\ & & \downarrow \quad \pi_\eta \\ & & \mathbb{T}^m \end{array} \tag{22}$$

Let $I_1,\dots,I_r$, be the basic Ad_G^*-invariant polynomials on $\mathfrak{g}^*$ and $P_1,\dots,P_{n-r}$ be linear functions on $\mathfrak{g}^*$ such that P_k, I_α are independent at η. Also, let $i_\alpha = I_\alpha\circ\Phi$ and $p_k = P_k\circ\Phi$ be the pull-backs of I_α and P_k by the momentum mapping, $\alpha = 1,\dots,r$, $k=1,\dots,n-r$.

Suppose that $f_1,\dots,f_m$ can be extended to commuting G-invariant functions in some G-invariant neighborhood V of $\mathcal{M}_c^o$. Then, within V, $\mathcal{M}_c^o$ is given by the equations

$$f_1 = c_1,\dots,f_m = c_m,\ i_1 = I_1(\eta),\dots,i_r = I_r(\eta),\ p_1 = P_1(\eta),\dots,P_{n-4}(\eta)\,.$$

From the Noether theorem the functions i_α, p_k commute with all G-invariant functions on M and, since i_α are G-invariant, the following commuting relations hold on V:

$$\begin{gathered}\{f_a, f_b\} = \{f_a, i_\alpha\} = \{i_\alpha, i_\beta\} = \{f_a, p_k\} = \{i_\alpha, p_k\} = 0,\\ a,b = 1,\dots,m, \quad \alpha,\beta = 1,\dots,r, \quad k = 1,\dots,n-r\,.\end{gathered} \tag{23}$$

Now, as in the case of non-commutative integrability of Hamiltonian systems [13, 23], $\mathcal{M}_c^o$ is a torus with tangent space spanned by X_{f_a}, X_{i_α}. Namely, from the

vanishing of the Poisson brackets (23) we get that the vector fields X_{f_a}, X_{i_α} are tangential to $\mathcal{M}_c^o$. Since they are independent, from the dimensional reasons, they span the tangent spaces $T_x\mathcal{M}_c^o$, $x \in \mathcal{M}_c^o$. Taking into account (23) and the relations $\omega(X_f, X_g) = -\{f, g\}$, $[X_f, X_g] = X_{\{f,g\}}$, $f, g \in C^\infty(M)$, we (re)obtain that $\mathcal{M}_c^o$ is isotropic. Furthermore the vector fields X_{f_a}, X_{i_α} commute between themselves. Since $\mathcal{M}_c^o$ is a compact manifold admitting $m+r = \dim \mathcal{M}_c^o$ independent commuting vector fields it is a $(m+r)$-dimensional torus, i.e., the bundle (22) is trivial.

In general, the flow of $\dot x = X_h$ over the torus $\mathcal{M}_c^o$ is not quasi-periodic: the vector field X_h do not commute with vector fields $X_{f_1}, \dots, X_{f_m}, X_{i_1}, \dots, X_{i_r}$ (although Poisson brackets $\{h, f_a\}$, $\{h, i_\alpha\}$ vanish on $\mathcal{M}_c^o$). $\Box$

Remark 2. Let $x \in \mathcal{M}^o$ and $z = \pi(x)$, where $\pi : M \to M/G$ is the canonical projection. By the use of (13) and the existence of local canonical coordinates on the Poisson manifold M/G within some neighborhood U of z (e.g, see [18]), the functions f_i can be always extended to G-invariant commuting functions in the G-invariant neighborhood $V = \pi^{-1}(U)$. Thus, if $\mathcal{M}^o$ is "small enough", it is a torus.

Remark 3. It is clear that complete commutative integrability of the reduced system in Theorem 2, can be replaced by the condition of non-commutative integrabilty.

Remark 4. The Hamiltonian flow (11) preserves the canonical measure $\Omega = \omega^{\dim M/2}$ (the Liouville theorem). If M_η is an invariant manifold of the equations (11), then the functions (17) are particular integrals. Therefore the time derivative of ϕ_α is of the form

$$\dot\phi_\alpha(x) = \{\phi_\alpha, h\} = \sum_\beta \psi_{\alpha\beta}(x)(\phi_\alpha - c_\alpha),$$

where $\psi_{\alpha\beta}$ are smooth functions. Let $\operatorname{tr}\psi$ be the trace of the matrix $\psi_{\alpha\beta}$. After straightforward calculations, one can prove that the flow (11) restricted to the invariant manifold M_η preserve the restriction of Ω to M_η if and only if

$$\operatorname{tr}\psi(x) = 0,$$

for $x \in M_\eta$. In particular, if the Hamiltonian h is a G-invariant function, then $\Omega|_{M_\eta}$ is an invariant volume form.

3. Partial Lagrange-Routh Reductions

Let (Q, l) be a natural mechanical system with Lagrangian $l = \frac{1}{2}(\kappa_q \dot q, \dot q) - v(q)$, where the metric κ is also regarded as a mapping $\kappa : TQ \to T^*Q$. The motion of the system is described by the Euler-Lagrange equations

$$\frac{\partial l}{\partial q} - \frac{d}{dt}\frac{\partial l}{\partial \dot q} = 0 \tag{24}$$

or by the Hamiltonian equations on the cotangent bundle T^*Q with the Hamiltonian $h(q, p) = \frac{1}{2}(p, \kappa^{-1}p) + v(q)$ being the Legendre transformation of l.

Let G be a compact connected Lie group acting freely on Q and $\pi : Q \to B = Q/G$ be the canonical projection. The G-action can be naturally extended to the Hamiltonian action on T^*Q: $g \cdot (q, p) = (g \cdot q, (dg^{-1})^* p)$ with the momentum mapping Φ given by (e.g, see [18])

$$(\Phi(q, p), \xi) = (p, \xi_q), \quad \xi \in \mathfrak{g}.$$

Here ξ_q is the vector given by the action of one-parameter subgroup $\exp(t\xi)$ at q.

For Lagrangian systems, it is convenient to work with tangent bundle reductions. Let $\mathcal{V}_q = \{\xi_q \mid \xi \in \mathfrak{g}\}$ be the tangent space to the fibber $G \cdot q$ (*vertical space at* q) and $\mathcal{V} = \cup_q \mathcal{V}_q$ be the vertical distribution. Then

$$(T^*Q)_0 = \Phi^{-1}(0) = \cup_q \operatorname{ann} \mathcal{V}_q, \qquad \operatorname{ann} \mathcal{V}_q = \{p \in T^*_q Q \mid (p, \xi_q) = 0, \, \xi \in \mathfrak{g}\}.$$

Consider the *horizontal distribution* $\mathcal{H} = \cup_q \mathcal{H}_q \subset TQ$ orthogonal to $\mathcal{V}$ with respect to the metric κ. Equivalently, $\mathcal{H}$ is the zero level-set of the tangent bundle momentum mapping Φ_l:

$$\mathcal{H} = \Phi_l^{-1}(0), \quad (\Phi_l(q, \dot q)\,, \xi) = \left(\frac{\partial l}{\partial \dot q}\,, \xi_q\right) = (\kappa_q \dot q\,, \xi_q), \quad \xi \in \mathfrak{g}. \tag{25}$$

Since $\kappa_q(\mathcal{H}_q) = \operatorname{ann} \mathcal{V}_q$, we see that $\mathcal{H}$ is invariant with respect to the "twisted" G-action

$$g \diamond (q, \, X) = (g \cdot q, \, \kappa_{g\cdot q}^{-1} \circ (dg^{-1})^* \circ \kappa_q(X)), \quad X \in T_q Q, \tag{26}$$

that is the pull-back of canonical symplectic G-action on T^*Q via metric κ:

$$\begin{array}{ccccc} & TQ & \xrightarrow{\kappa} & T^*Q & \\ g\diamond & \downarrow & & \downarrow & g\cdot \\ & TQ & \xrightarrow[\kappa]{} & T^*Q & \end{array}$$

From Theorem 1 we obtain

Theorem 3. (i) *The horizontal distribution (25) is an invariant submanifold of the Euler-Lagrange equations (24) if and only if the potential* v *and the restriction* $\kappa_{\mathcal{H}}$ *of the metric* κ *to* $\mathcal{H}$ *are* G*-invariant with respect to the action (26).*

(ii) *If* $\mathcal{H}$ *is an invariant submanifold of the system* (Q, l) *then the trajectories* $q(t)$ *of the natural mechanical system with velocities* $q(t)$ *that belong to* $\mathcal{H}$ *project to the trajectories* $b(t) = \pi(q(t))$ *of the natural mechanical system* (B, L) *with the potential* $V(\pi(q)) = v(q)$ *and the metric* K *obtained from* $\kappa_{\mathcal{H}}$ *via identification* $\mathcal{H}/G \approx TB$.

Note that when κ is G-invariant, the twisted G-action (26) coincides with usual G-action: $g \cdot (q, X) = (g \cdot q, dg(X))$ and the induced metric K is the submersion metric. In this case Theorem 3 is exactly the classical method of E. J. Routh for eliminating cyclic coordinates [27, 2, 21].

Definition 2. By the analogy with the Lagrange-Routh reduction we shall call the procedure of passing from the Lagrangian system (Q, l) to the system (B, L) a *partial Lagrange-Routh reduction*.

Remark 5. The distribution (25), in general, is nonintegrable. One can interpret the restriction of the Lagrangian system (Q, l) to $\mathcal{H}$ as a nonholonomic system $(Q, l, \mathcal{H})$ with a property that the reaction forces are equal to zero.

3.1. **Geodesic Flows on Compact Lie Groups.** Let G be a compact Lie group, $\mathfrak{g}$ be the Lie algebra of G and $\langle \cdot, \cdot \rangle$ be a Ad_G-invariant scalar product on $\mathfrak{g}$. Let $a \in \mathfrak{g}$ be an arbitrary element and let b belongs to the center of $\mathfrak{g}_a$. Here $\mathfrak{g}_a = \{\eta \in \mathfrak{g}, [a, \eta] = 0\}$ is the isotropy algebra of a. Let $\mathfrak{g} = \mathfrak{g}_a + \mathfrak{d}$ be the orthogonal

decomposition. Consider the linear operator (so called *sectional operators* [13]) $\mathrm{A}_{a,b,\mathrm{C}}:\mathfrak{g}\to\mathfrak{g}$, defined by

$$\mathrm{A}_{a,b,\mathrm{C}}(\xi)=\mathrm{ad}_a^{-1}\circ\mathrm{ad}_b\circ\mathrm{pr}_{\mathfrak{d}}(\xi)+\mathrm{C}(\mathrm{pr}_{\mathfrak{g}_a}\xi),$$

where $\mathrm{pr}_{\mathfrak{d}}$ and $\mathrm{pr}_{\mathfrak{g}_a}$ are the orthogonal (with respect to $\langle\cdot,\cdot\rangle$) projections to $\mathfrak{d}$ and $\mathfrak{g}_a$, respectively and $\mathrm{C}:\mathfrak{g}_a\to\mathfrak{g}_a$ is symmetric. We can always find b and C such that $\mathrm{A}_{a,b,\mathrm{C}}$ is positive definite.

Identify $\mathfrak{g}^*$ with $\mathfrak{g}$ by means of the scalar product $\langle\cdot,\cdot\rangle$ and consider the left-trivialization:

$$T^*G\approx_l G\times\mathfrak{g}=\{(g,\xi)\}.$$

Then the quadratic form

$$h_{a,b,\mathrm{C}}=\frac{1}{2}\langle\mathrm{A}_{a,b,\mathrm{C}}(\xi),\xi\rangle \tag{27}$$

can be regarded as the Hamiltonian of a left-invariant Riemannian metric on G. Denote this metric by $\kappa_{a,b,\mathrm{C}}$.

Let G_a be the adjoint isotropy group of the element a and consider the *right* G_a-action on G. With the above notation, the momentum mapping and its zero level-set are given by

$$\Phi(g,\xi)=\mathrm{pr}_{\mathfrak{g}_a}(\xi), \tag{28}$$

$$(T^*G)_0\approx_l G\times\mathfrak{d}\,. \tag{29}$$

Lemma 4. *The metric $\kappa_{a,b,\mathrm{C}}$ is invariant with respect to the G_a-action, i.e., the momentum map (28) is preserved along the geodesic flow*

$$\dot\xi=[\xi,\mathrm{A}_{a,b,\mathrm{C}}(\xi)],\quad \dot g=g\cdot\mathrm{A}_{a,b,\mathrm{C}}(\xi) \tag{30}$$

if and only if the quadratic form $\langle\xi,\mathrm{C}(\xi)\rangle$ is Ad_{G_a}-invariant:

$$[\xi,\mathrm{C}(\xi)]=0,\quad \xi\in\mathfrak{g}_a. \tag{31}$$

If a is regular element of the Lie algebra $\mathfrak{g}$, then $\mathfrak{g}_a$ is Abelian and the condition (31) is satisfied for an arbitrary operator C.

Suppose (31) holds. Then we can project the metric $\kappa_{a,b,\mathrm{C}}$ to the homogeneous space G/G_a, that is to the adjoint orbit of a: $G/G_a\approx O(a)=\{\mathrm{Ad}_g(a)\,|\,g\in G\}$. Since we deal with the right action, the vertical distribution is left-invariant: $\mathcal{V}_g=g\cdot\mathfrak{g}_a$, while from the definition of $\kappa_{a,b,\mathrm{C}}$, the horizontal distribution is $\mathcal{H}_g=g\cdot\mathfrak{d}$ and the submersion metric does not depend on C.

Denote the submersion metric by $K_{a,b}$. The cotangent bundle $T^*\mathcal{O}(a)$ can be realized as a submanifold of $\mathfrak{g}\times\mathfrak{g}$

$$T^*\mathcal{O}(a)=\{(x,p)\,|\,x=\mathrm{Ad}_g(a),p\in\mathfrak{g}_x^\perp\}, \tag{32}$$

with the pairing between $p\in T_x^*\mathcal{O}(a)$ and $\eta\in T_x\mathcal{O}(a)$ given by $p(\eta)=\langle p,\eta\rangle$. Let $b_x=\mathrm{Ad}_g b$. Then the Hamiltonian and the geodesic flow for the metric $K_{a,b}$ in redundant variables (x,p) are given by (see [6])

$$H_{a,b}(x,p)=\frac{1}{2}\langle\mathrm{ad}_{b_x}p,\mathrm{ad}_x p\rangle=-\frac{1}{2}\langle\mathrm{ad}_x\mathrm{ad}_{b_x}p,p\rangle, \tag{33}$$

$$\dot x=-\mathrm{ad}_x\mathrm{ad}_{b_x}p=[[b_x,p],x], \tag{34}$$

$$\dot p=-\mathrm{ad}_x^{-1}[p,[x,[b_x,p]]]+\mathrm{pr}_{\mathfrak{g}_x}[[b_x,p],p]. \tag{35}$$

Now, let us perturb the metric $\kappa_{a,b,\mathrm{C}}$ as follows. Take $\delta = \langle \mathrm{B}_\delta(\mathrm{pr}_{\mathfrak{d}}\,\xi), \xi\rangle + \frac{1}{2}\langle \mathrm{C}_\delta(\mathrm{pr}_{\mathfrak{g}_a}\,\xi), \xi\rangle$, $\mathrm{B}_\delta : \mathfrak{d} \to \mathfrak{g}_a$, $\mathrm{C}_\delta : \mathfrak{g}_a \to \mathfrak{g}_a$, such that

$$h_\delta(g, \xi) = h_{a,b,\mathrm{C}} + \delta = \frac{1}{2}\langle \mathrm{A}_\delta \xi, \xi\rangle \tag{36}$$

is positive definite. Then (36) will be the Hamiltonian function of the left-invariant metric that we shall denote by κ_δ. Since $h_\delta = h|_{(T^*G)_0}$, the geodesic flow of κ_δ

$$\dot{\xi} = [\xi, \mathrm{A}_\delta(\xi)], \quad \dot{g} = g \cdot \mathrm{A}_\delta(\xi). \tag{37}$$

has the invariant relation (29). Hence we can perform the partial reduction. The metrics κ_δ and $\kappa_{a,b,\mathrm{C}}$ induce the same metric $K_{a,b}$ on the orbit $O(a)$, but their horizontal distributions $\mathcal{H}^\delta$ and $\mathcal{H}$ are different for $\mathrm{B} \neq 0$:

$$\mathcal{H}^\delta_g = \kappa_\delta^{-1}(g \cdot \mathfrak{d}) \neq g \cdot \mathfrak{d} = \mathcal{H}_g\,.$$

In the case when a is a singular element of $\mathfrak{g}$ we can take $\delta = \frac{1}{2}\langle \mathrm{C}_\delta(\mathrm{pr}_{\mathfrak{g}_a}\,\xi), \xi\rangle$ such that δ is not Ad_{G_a}-invariant. Then the perturbed metric has the same horizontal distribution as the non-perturbed one: $\mathcal{H}^\delta_g = g \cdot \mathfrak{d}$.

The geodesic flow (34), (35) is completely integrable in the non-commutative sense. Moreover, the system is also integrable in the usual commutative sense by means of analytic functions, polynomial in momenta and which can be lifted to commuting G_a-invariant functions on T^*G (see [5, 22, 6]). Thus, according Theorem 2 we obtain

Corollary 5. *The equations (30) and (37) have the same invariant isotropic foliation of (29).*

3.2. **Local Description.** To clear up the difference between geodesic flows (30) and (37), let us write down the problem in local coordinates.

Suppose a is a regular element of $\mathfrak{g}$. Then $G_a \approx \mathbb{T}^n$ ia a maximal torus. Locally, we have $T^*G \approx T^*O(a) \times T^*\mathbb{T}^n$. We take coordinates (q, φ, p, ϕ) in T^*G such that

$$(q, p) = (q_1, \dots, q_m, p_1, \dots, p_m) \quad \text{and} \quad (\varphi, \phi) = (\varphi_1, \dots, \varphi_n, \phi_1, \dots, \phi_n)$$

are canonical local coordinates on $T^*O(a)$ and $T^*\mathbb{T}^n$. Locally, the right $\mathbb{T}^n$-action is given by translations in φ coordinates and $(T^*G)_0 = \{(q, \varphi, p, 0)\}$.

The Hamiltonians $h_{a,b,\mathrm{C}}$ and h_δ are of the forms:

$$h_{a,b,\mathrm{C}} = \frac{1}{2}\sum A^{ij}p_ip_j + \sum B^{i\alpha}p_i\phi_\alpha + \frac{1}{2}\sum C^{\alpha\beta}\phi_\alpha\phi_\beta,$$
$$h_\delta = \frac{1}{2}\sum A^{ij}p_ip_j + \sum B^{i\alpha}_\delta p_i\phi_\alpha + \frac{1}{2}\sum C^{\alpha\beta}_\delta\phi_\alpha\phi_\beta,$$

where A^{ij}, $B^{i\alpha}$, $C^{\alpha\beta}$ do not depend on the variables φ_α.

The partially reduced system

$$\dot{q}_i = \frac{\partial H_{a,b}}{\partial p_i} = \sum A^{ij}p_j, \quad \dot{p}_i = -\frac{\partial H_{a,b}}{\partial q_i} = -\sum \frac{\partial A^{kj}}{\partial q_i}p_kp_j, \quad i = 1, \dots, m \tag{38}$$

is completely integrable and we can treat $q(t)$ and $p(t)$ as known functions of time. Here $H_{a,b} = \frac{1}{2}\sum A^{ij}p_ip_j$ is the reduced Hamiltonian (33).

The equations of the geodesic flows (30) and (37) on (29) in variables q_i, p_i are given by (38). The equations in variables φ_α, respectively, are given by:

$$\dot{\varphi}_\alpha = \sum B^{i\alpha}(q)p_i \quad \text{and} \quad \dot{\varphi}_\alpha = \sum B^{i\alpha}_\delta(q, \varphi)p_i, \quad \alpha = 1, \dots, n.$$

While the first system is solvable, the second one, generically, is not. Globally, (29) is foliated on invariant tori with quasi-periodic and non-quasi-periodic flows of (30) and (37), respectively.

4. n-Dimensional Hess-Appel'rot System

Consider the motion of an n-dimensional rigid body around a fixed point $O = (0, 0, \dots, 0)$ in the n-dimensional Euclidean vector space $(\mathbb{R}^n, (\cdot, \cdot))$. The configuration space of the system is the Lie group $SO(n)$: the element $g \in SO(n)$ maps the moving coordinate system (attached to the body) to the fixed one (e.g., see [12]). Let $F_1, \dots, F_n$, $E_1, \dots, E_n$ and $f_1, \dots, f_n$, $e_1, \dots, e_n$ be the orthonormal bases attached to the body and fixed in the space regarded in the space frame and moving frame:

$$
\begin{aligned}
&E_1 = f_1 = (1, 0, \dots, 0, 0)^T, \dots, E_n = f_n = (0, 0, \dots, 0, 1)^T, \\
&E_1 = g \cdot e_1, \dots, E_n = g \cdot e_n, \quad F_1 = g \cdot f_1, \dots, F_n = g \cdot f_n.
\end{aligned}
$$

We can consider the components of the vectors $e_1, \dots, e_n$ (or $F_1, \dots, F_n$) as redundant coordinates on $SO(n)$.

For a path $g(t) \in SO(n)$, *the angular velocity in the body frame* and *angular velocity in the space frame* are defined by $\omega(t) = g^{-1} \cdot \dot g(t) \in so(n)$ and $\Omega(t) = \dot g \cdot g^{-1} = g \cdot \omega \cdot g^{-1} = \mathrm{Ad}_g\, \omega$, respectively. From the conditions $0 = \dot E_i = \dot g \cdot e_i + g \cdot \dot e_i$ and $\dot F_i = \dot g \cdot f_i$, the vectors $e_1, \dots, e_n$ and $F_1, \dots, F_n$ satisfy Poisson equations

$$
\dot e_i = -\omega \cdot e_i, \quad \dot F_i = \Omega \cdot F_i, \quad i = 1, \dots, n.
$$

The kinetic energy of a rigid body is a left-invariant quadratic form $\frac{1}{2}\langle \mathrm{I}\, \omega, \omega \rangle$, where $\mathrm{I} : so(n) \to so(n)$ is a non-degenerate *inertia operator* and $\langle X, Y \rangle = -\frac{1}{2}\mathrm{tr}(XY)$ denotes the Killing metric on $so(n)$. For a "physical" rigid body, $\mathrm{I}\, \omega$ has the form $I\omega + \omega I$, where I is a symmetric $n \times n$ matrix [12]. We will relax this condition, considering an arbitrary positive definite operator.

Further, suppose that the body is placed in the homogeneous gravitational force field in the direction e_n and the position of the center of mass of a rigid body is ρf_n. Then the potential is $v = \rho \mathfrak{G}\mathfrak{M}\,(f_n, e_n)$, where $\mathfrak{G}$ is the gravitational constant and $\mathfrak{M}$ is the mass of the body. Let $m = \mathrm{I}\, \omega$ be the *angular momentum in the body frame* and $\mathrm{A} = \mathrm{I}^{-1}$. By the use of Killing metric we can identify $so(n)$ and $so(n)^*$. Then the Hamiltonian in the left-trivialization

$$
T^*SO(n) \approx_l SO(n) \times so(n) = \{(g, m)\} \tag{39}
$$

reads

$$
h = \frac{1}{2}\langle m, \mathrm{A} m \rangle + \rho \mathfrak{G}\mathfrak{M}\,(f_n, e_n)
$$

and the equations of the system, in redundant variables $(e_1, \dots, e_n, m)$, take the form of the Euler-Poisson equations

$$
\dot m = [m, \omega] + \rho \mathfrak{G}\mathfrak{M}\, f_n \wedge e_n, \qquad \omega = \mathrm{A} m \tag{40}
$$

$$
\dot e_i = -\omega \cdot e_i, \quad i = 1, \dots, n. \tag{41}
$$

4.1. **Invariant Relations.** Consider the orthogonal, symmetric pair decomposition $\mathfrak{k}+\mathfrak{d}$ of the Lie algebra $so(n)$:

$$\mathfrak{k}=\operatorname{span}\{f_i\wedge f_j,\, 1\le i<j\le n-1\}\cong so(n-1), \quad \mathfrak{d}=\operatorname{span}\{f_i\wedge f_n,\, 1\le i\le n-1\}.$$

Then we can write h as

$$h=\frac{1}{2}\langle m_{\mathfrak{k}},\mathrm{A}_{\mathfrak{k}}m_{\mathfrak{k}}\rangle+\langle m_{\mathfrak{k}},\mathrm{B}m_{\mathfrak{d}}\rangle+\frac{1}{2}\langle m_{\mathfrak{d}},\mathrm{A}_{\mathfrak{d}}m_{\mathfrak{d}}\rangle+\rho\mathfrak{GM}\,(f_n,e_n),$$

where $\mathrm{A}_{\mathfrak{k}}=\operatorname{pr}_{\mathfrak{k}}\circ\mathrm{A}\circ\operatorname{pr}_{\mathfrak{k}}$, $\mathrm{A}_{\mathfrak{d}}=\operatorname{pr}_{\mathfrak{d}}\circ\mathrm{A}\circ\operatorname{pr}_{\mathfrak{d}}$, $\mathrm{B}=\operatorname{pr}_{\mathfrak{k}}\circ\mathrm{A}\circ\operatorname{pr}_{\mathfrak{d}}$, $m_{\mathfrak{k}}=\operatorname{pr}_{\mathfrak{k}}m$, $m_{\mathfrak{d}}=\operatorname{pr}_{\mathfrak{d}}m$.

Let $SO(n-1)$ be the subgroup of $SO(n)$ with the Lie algebra $\mathfrak{k}$. Consider the *right* $SO(n-1)$ action (rotations of a rigid body "around" the vector f_n). The momentum mapping and its zero-level set are given by $\Phi=m_{\mathfrak{k}}=\operatorname{pr}_{\mathfrak{k}}m$ and

$$(T^*SO(n))_0=\Phi^{-1}(0)\approx_l\{(g,m)\,|\,m_{\mathfrak{k}}=0\}=SO(n)\times\mathfrak{d}\,. \tag{42}$$

From the relations $[\mathfrak{k},\mathfrak{d}]\subset\mathfrak{d}$, $[\mathfrak{d},\mathfrak{d}]\subset\mathfrak{k}$, the orthogonal projections of equations (40) to $\mathfrak{k}$ and $\mathfrak{d}$ are given by

$$\dot m_{\mathfrak{k}}=[m_{\mathfrak{k}},\mathrm{A}_{\mathfrak{k}}m_{\mathfrak{k}}]+[m_{\mathfrak{k}},\mathrm{B}m_{\mathfrak{d}}]+[m_{\mathfrak{d}},\mathrm{A}_{\mathfrak{d}}m_{\mathfrak{d}}]+[m_{\mathfrak{d}},\mathrm{B}^Tm_{\mathfrak{k}}], \tag{43}$$

$$\dot m_{\mathfrak{d}}=[m_{\mathfrak{k}},\mathrm{A}_{\mathfrak{d}}m_{\mathfrak{d}}]+[m_{\mathfrak{k}},\mathrm{B}^Tm_{\mathfrak{k}}]+[m_{\mathfrak{d}},\mathrm{A}_{\mathfrak{k}}m_{\mathfrak{k}}]+[m_{\mathfrak{d}},\mathrm{B}m_{\mathfrak{d}}]+\rho\mathfrak{GM}\,f_n\wedge e_n. \tag{44}$$

Hence, (42) is an invariant submanifold if and only if $[m_{\mathfrak{d}},\mathrm{A}_{\mathfrak{d}}m_{\mathfrak{d}}]=0$, i.e., $\langle m_{\mathfrak{d}},\mathrm{A}_{\mathfrak{d}}m_{\mathfrak{d}}\rangle$ is the $\mathrm{Ad}_{SO(n-1)}$-invariant quadratic form on $\mathfrak{d}$. Since $SO(n)/SO(n-1)$ is a rank one symmetric space, $\mathrm{Ad}_{SO(n-1)}$-invariant quadratic form on $\mathfrak{d}$ is unique (up to multiplication by a constant). Thus, we come to the following proposition

Lemma 6. *The submanifold (42) is an invariant set of the Euler-Poisson equations (40), (41) if and only if*

$$\operatorname{pr}_{\mathfrak{d}}\circ\mathrm{A}\circ\operatorname{pr}_{\mathfrak{d}}=a\mathrm{Id}_{\mathfrak{d}}, \tag{45}$$

where $\mathrm{Id}_{\mathfrak{d}}$ *is the identity operator and* $a\in\mathbb{R}$.

Suppose (45) holds. In coordinates $m_{ij}=\langle m,f_i\wedge f_j\rangle$, the invariant set (42) is given by

$$m_{ij}=0,\qquad 1\le i<j\le n-1. \tag{46}$$

For $n=3$, after usual identification $(\mathbb{R}^3,\times)\cong(so(n),[\cdot,\cdot])$

$$\vec m\longleftrightarrow\begin{pmatrix}0&-m_3&m_2\\ m_3&0&-m_1\\ -m_2&m_1&0\end{pmatrix}, \tag{47}$$

the operator A take the form (6) and the invariant relation (46) become (7).

Therefore, it is natural to call the rigid body system (40), (41), where (45) holds, a *n-dimensional Hess-Appel'rot system.*

4.2. **Right Trivialization and Reduction to S^{n-1}.** In order to describe the partially reduced system we consider the problem in the right-trivialization

$$T^*SO(n)\approx_r SO(n)\times so(n)=\{(g,M)\},$$

i.e., in the frame fixed in the space. Instead of $(e_1,\dots,e_n)$ we use $(F_1,\dots,F_n)$ as redundant coordinates on $SO(n)$. Then the sphere $S^{n-1}=SO(n)/SO(n-1)$ can

be identified with the positions of the vector F_n:

$$\begin{array}{ccc} SO(n-1) & \longrightarrow & SO(n) \\ & & \downarrow \pi \\ & & S^{n-1} \end{array}, \qquad \pi(F_1,\dots,F_n) = F_n\,.$$

The equations of motion (40), (41) in the space frame take the form

$$\begin{aligned} \dot M &= \rho\mathfrak{G}\mathfrak{M}\, F_n \wedge E_n, \\ \dot F_i &= \Omega \cdot F_i, \quad i=1,\dots,n, \end{aligned} \tag{48}$$

where the angular momentum M and velocity Ω in the space are related to the angular momentum m and velocity ω in body coordinates via $M = \mathrm{Ad}_g\, m$ and $\Omega = \mathrm{Ad}_g\, \omega$, respectively. Therefore

$$\Omega = \mathrm{Ad}_g(\omega) = \mathrm{Ad}_g \circ \mathrm{A} \circ \mathrm{Ad}_{g^{-1}}(M). \tag{49}$$

Since $F_i \wedge F_j = \mathrm{Ad}_g(f_i \wedge f_j)$, the invariant submanifold (42) in the right-trivialization reads

$$(T^*SO(n))_0 \approx_r \{(g, \mathcal{D}_g),\, \mathcal{D}_g = \mathrm{Ad}_g(\mathfrak{d})\}.$$

Let $\mathcal{K}_g = \mathrm{Ad}_g(\mathfrak{k})$ and $X = X_{\mathcal{K}_g} + X_{\mathcal{D}_g}$ be the decomposition of $X \in so(n)$ with respect to $so(n) = \mathcal{K}_g + \mathcal{D}_g$. Note that the orthogonal projection to $\mathcal{D}_g$ given by

$$X_{\mathcal{D}_g} = (X \cdot F_n) \wedge F_n. \tag{50}$$

On $(T^*SO(n))_0$ we have $\omega = A_{\mathfrak{d}}(m) + \mathrm{B}(m) = a \cdot m + \mathrm{B}(m)$. Thus

$$\Omega = \Omega_{\mathcal{K}_g} + \Omega_{\mathcal{D}_g}, \quad \Omega_{\mathcal{K}_g} = \mathrm{Ad}_g \circ \mathrm{B} \circ \mathrm{Ad}_{g^{-1}}(M), \quad \Omega_{\mathcal{D}_g} = a \cdot M,$$

Furthermore, from (50) we have $\Omega \cdot F_n = \Omega_{\mathcal{D}_g} \cdot F_n$. Therefore, on $(T^*SO(n))_0$, the Hess-Appel'rot system form the closed system in variables $(F_n, \Omega_{\mathcal{D}_g})$:

$$\begin{aligned} \dot\Omega_{\mathcal{D}_g} &= a\rho\mathfrak{G}\mathfrak{M}\, F_n \wedge E_n, \\ \dot F_n &= \Omega_{\mathcal{D}_g} \cdot F_n, \end{aligned} \tag{51}$$

From the second equation we get

$$\Omega_{\mathcal{D}_g} = \dot F_n \wedge F_n, \quad \text{i.e.,} \quad M = \frac{1}{a} \dot F_n \wedge F_n \tag{52}$$

and the first equation can be rewritten in the form:

$$\ddot F_n = -a\rho\mathfrak{G}\mathfrak{M}\, E_n + \mu F_n\,. \tag{53}$$

Here the Lagrange multiplier μ is determined from the condition $(F_n, F_n) = 1$.

Therefore we obtain:

Theorem 7. *The motion of the center of the mass of the n-dimensional Hess-Appel'rot system (40), (41), (42), (45) in the space frame is described by the spherical pendulum equations (53).*

The same statement, for the classical Hess-Appel'rot system is proved by Zhukovski (see [29, 8]).

It is well known that the partially reduced system (53) is completely integrable in the non-commutative sense. The tangent bundle TS^{n-1} is foliated by two-dimensional invariant manifolds that are level-sets of the energy and integrals

$$\langle \dot F_n \wedge F_n, E_i \wedge E_j \rangle = (\dot F_n, E_i)(F_n, E_j) - (\dot F_n, E_j)(F_n, E_i), \quad 1 \le i < j \le n-1.$$

Using (52), we see that the lifting of these integrals to $(T^*SO(n))_0$ are components of the momentum map of the *left* $SO(n-1)$-action:

$$\mathcal{G}_{ij} = a \cdot \langle M, E_i \wedge E_j \rangle, \quad 1 \le i < j \le n-1 \tag{54}$$

(note that (54) are integrals on the whole phase space $T^*SO(n)$ as well). Whence

Corollary 8. *The invariant set (42) of the Hess-Appel'rot system is almost everywhere foliated by* $(2 + (n-1)(n-2)/2)$*-dimensional isotropic invariant manifolds, level sets of the Hamiltonian function and integrals (54).*

The above considerations motivate us for the following definition

Definition 3. We shall say that a Hamiltonian system (11) satisfies *geometrical Hess-Appel'rot conditions* if it has an invariant relation (9) and the partially reduced system (15) is completely integrable.

The geodesic flows on compact Lie groups presented in Section 3 provide examples of systems that satisfy geometrical Hess-Appel'rot conditions. Note that the condition that the partially reduced system is integrable differs from the notion of *restricted integrability* given in [11].

4.3. **Reduction to** $(so(n) \times \mathbb{R}^n)^*$**.** The system (40), (41) is always *left* $SO(n-1)$-invariant (rotations of a rigid body "around" the vector e_n). Denote $\gamma = e_n$, $r = \rho f_n$. The equations (40) together with the last Poisson equation,

$$\dot{m} = [m, \omega] + \mathfrak{G}\mathfrak{M}\, r \wedge \gamma, \quad \dot{\gamma} = -\omega \cdot \gamma, \quad \omega = \mathrm{A}m, \tag{55}$$

can be seen as a left $SO(n-1)$-reduction of (40), (41). This is a Hamiltonian system on the dual space of Lie algebra $e(n) = so(n) \times \mathbb{R}^n$ with respect to the usual Lie-Poisson bracket. For $n = 3$, after identification (47), the equations (55) get the familiar form (1).

Now suppose the Hess-Appel'rot condition (45) is satisfied. Then the Euler-Poisson equations (55) on the invariant set (46) coordinately read:

$$\begin{aligned}
\dot{m}_{in} &= -\sum_{j=1}^{n-1} \omega_{ij} m_{jn} - \rho \mathfrak{G}\mathfrak{M}\, \gamma_i, \quad i = 1, \dots, n-1, \\
\dot{\gamma}_i &= -a\gamma_n m_{in} - \sum_{j=1}^{n-1} \omega_{ij} \gamma_j, \quad i = 1, \dots, n-1, \\
\dot{\gamma}_n &= a \sum_{j=1}^{n-1} \gamma_j m_{jn},
\end{aligned} \tag{56}$$

where $\omega_{ij} = \langle \mathrm{B}m, f_i \wedge f_j \rangle$, $1 \le i < j \le n-1$. The equations (56) always have three integrals

$$\begin{aligned}
\mathcal{F}_1 &= \frac{a}{2} \sum_{i=1}^{n-1} m_{in}^2 + \rho \mathfrak{G}\mathfrak{M}\, \gamma_n, \quad \mathcal{F}_2 = \sum_{i=1}^{n} \gamma_i^2 = 1, \\
\mathcal{F}_3 &= \sqrt{\sum_{1 \le i < j \le n-1} (m_{in}\gamma_j - m_{jn}\gamma_i)^2},
\end{aligned} \tag{57}$$

which correspond to integrals (2). As we sow above, the variable $\gamma_n = (e_n, f_n) = (E_n, F_n)$ satisfies the vertical component equation for the motion of the spherical pendulum and can be found by quadratures.

In general, equations (56) have no smooth invariant measure. The non-existence of a smooth invariant measure is the reflection of the non-solvability by quadratures of the classical system (1), (3), (4). Namely, by the Euler-Jacobi theorem (e.g., see [2], page 131), the invariant measure together with the integrals (2), (4) would implies solvability of the system by quadratures.

Example 1. As an illustration, consider the case $n = 4$ and Hamiltonian

$$\begin{aligned} h &= \frac{1}{2}(a_1 m_{23}^2 + a_2 m_{13}^2 + a_3 m_{12}^2) + \frac{a}{2}(m_{14}^2 + m_{24}^2 + m_{34}^2) + \\ &+ b_1 m_{12} m_{14} + b_2 m_{12} m_{24} + b_3 m_{12} m_{34} + \rho \mathfrak{G}\mathfrak{M}\gamma_n. \end{aligned}$$

Then the equations in variables m_{12}, m_{23}, m_{13} are

$$\begin{aligned} &\dot{m}_{12} = m_{13} m_{23}(a_2 - a_1) + m_{12}(b_1 m_{24} - b_2 m_{14}), \\ &\dot{m}_{23} = m_{12} m_{13}(a_3 - a_2) + m_{12}(b_2 m_{34} - b_3 m_{24}) + m_{13}(b_1 m_{14} + b_2 m_{24} + b_3 m_{34}), \\ &\dot{m}_{13} = m_{23} m_{12}(a_1 - a_3) + m_{12}(b_1 m_{34} - b_3 m_{14}) - m_{23}(b_1 m_{14} + b_2 m_{24} + b_3 m_{34}), \end{aligned}$$

while the equations in other variables, on the invariant set

$$m_{12} = m_{13} = m_{23} = 0, \tag{58}$$

take the form

$$\begin{aligned} &\dot{m}_{14} = -m_{24}(b_1 m_{14} + b_2 m_{24} + b_3 m_{34}) - \rho \mathfrak{G}\mathfrak{M}\gamma_1, \\ &\dot{m}_{24} = m_{14}(b_1 m_{14} + b_2 m_{24} + b_3 m_{34}) - \rho \mathfrak{G}\mathfrak{M}\gamma_2, \\ &\dot{m}_{34} = -\rho \mathfrak{G}\mathfrak{M}\gamma_3, \\ &\dot{\gamma}_1 = -a\gamma_4 m_{14} - \gamma_2(b_1 m_{14} + b_2 m_{24} + b_3 m_{34}), \\ &\dot{\gamma}_2 = -a\gamma_4 m_{24} + \gamma_1(b_1 m_{14} + b_2 m_{24} + b_3 m_{34}), \\ &\dot{\gamma}_1 = -a\gamma_4 m_{34}, \\ &\dot{\gamma}_4 = a(\gamma_1 m_{14} + \gamma_2 m_{24} + \gamma_3 m_{34}). \end{aligned} \tag{59}$$

Together with (57) equations (59) have a supplementary integral

$$\mathcal{F}_{12} = m_{14}\gamma_2 - m_{24}\gamma_1, \tag{60}$$

implying the foliation on 3-dimensional invariant manifolds of (58).

In redundant coordinates (F_1, F_2, F_3, F_4, M) the integral (60) on $T^*SO(4)$ reads

$$\begin{aligned} \mathcal{F}_{12} &= \langle m, f_1 \wedge f_4\rangle(e_4, f_2) - \langle m, f_2 \wedge f_4\rangle(e_4, f_1) \\ &= \langle M, F_1 \wedge F_4\rangle(E_4, F_2) - \langle M, F_2 \wedge F_4\rangle(E_4, F_1)\,. \end{aligned} \tag{61}$$

It is clear that $\mathcal{F}_{12}$ is independent of integrals (54). Whence the 5-dimensional invariant isotropic manifolds outlined in Corollary 8 are foliated on 4-dimensional invariant level-sets of (61).

Note that the system (59) has an invariant measure if and only if $b_1 = b_2 = 0$. Also, recall that for $a_1 = a_2 = a_3 \neq a$, $b_1 = b_2 = b_3 = 0$, the Euler-Poisson equations represent the motion of a dynamically symmetric heavy rigid body, 4-dimensional version of the Lagrange top [3].

Therefore, if $b_1 = b_2 = b_3 = 0$ and $a_1 \neq a_2 \neq a_3$ then the equations (59) coincides to the 4-dimensional Lagrange top equations restricted to the invariant

level-set (58), although the systems are different on the whole phase space. They have additional integrals

$$
\begin{aligned}
&\mathcal{F}_{13} = m_{14}\gamma_3 - m_{34}\gamma_1 \;\left(= \langle M, F_1 \wedge F_3\rangle (E_4, F_3) - \langle M, F_3 \wedge F_4\rangle (E_4, F_1)\right), \\
&\mathcal{F}_{23} = m_{24}\gamma_3 - m_{34}\gamma_2 \;\left(= \langle M, F_2 \wedge F_4\rangle (E_4, F_3) - \langle M, F_3 \wedge F_4\rangle (E_4, F_2)\right). \qquad (62)
\end{aligned}
$$

Since the equations (59) preserve the standard measure in variables m_{14}, m_{24}, m_{34}, $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ and the invariant manifolds given by integrals (57), (60), (62) are two-dimensional, by the Euler-Jacobi theorem, they are solvable by quadratures. On the other side, the Hamiltonian and functions (54), (61), (62) provide 3-dimensional invariant foliation of 9-dimensional manifold $(T^*SO(4))_0$.

4.4. **L-A Pair.** The non-commutative integrability of the n-dimensional Lagrange top is proved by Beljaev [3] and the L-A pair is given by Reyman and Semenov-Tian-Shansky [26]. The L-A pair of the similar form can be used for the Hess-Appel'rot system on $so(n) \times \mathbb{R}^n$. Let m^*, ω^*, γ^*, r^* be the $so(n+1)$-matrixes given by

$$
m^* = \begin{pmatrix} m & 0 \\ 0 & 0 \end{pmatrix}, \quad \gamma^* = \begin{pmatrix} \mathbf{0} & \gamma \\ -\gamma^t & 0 \end{pmatrix},
$$
$$
\omega^* = \begin{pmatrix} \omega & 0 \\ 0 & 0 \end{pmatrix}, \quad r^* = \begin{pmatrix} \mathbf{0} & \rho f_n \\ -\rho f_n^t & 0 \end{pmatrix},
$$

where $\mathbf{0}$ is the zero $n \times n$ matrix.

Then, under the Hess-Appel'rot conditions (45) and (46), the Euler-Poisson equations (55) are equivalent to the matrix equation with a spectral parameter λ

$$
\dot{L}(\lambda) = [L(\lambda), A(\lambda)], \qquad (63)
$$

where $L(\lambda) = \gamma^* + \lambda m^* + \lambda^2 \frac{1}{a}\mathfrak{G}\mathfrak{M} r^*$ and $A(\lambda) = \omega^* + \lambda \mathfrak{G}\mathfrak{M} r^*$.

Indeed, the straightforward computations shows that the terms with λ^0 and λ^1 in (63) are equivalent to the equations (55). The left hand side term with λ^2 is identically equal to zero. It can be proved that the right hand side term with λ^2 vanish, on the invariant submanifold (46), if and only if (45) holds.

On the invariant set (46) the spectral curve is

$$
\begin{aligned}
&p(\lambda, \mu) = \det(L(\lambda) - \mu \mathrm{Id}) = (-\mu)^{n-3}\left(\mu^4 + \mu^2 P(\lambda) + Q(\lambda)^2\right) = 0, \qquad (64)\\
&P(\lambda) = \mathcal{F}_2 + \frac{2}{a}\lambda^2 \mathcal{F}_1 + \lambda^4 \left(\frac{\rho \mathfrak{G}\mathfrak{M}}{a}\right)^2, \quad Q(\lambda) = \lambda \mathcal{F}_3,
\end{aligned}
$$

where $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ are integrals (57). The curve (64) is reducible (for $n > 3$) and consists of the $n-3$ copies of the rational curve $\mu = 0$ and the singular algebraic curve

$$
\Gamma: \quad \mu^4 + \mu^2 P(\lambda) + Q(\lambda)^2 = 0. \qquad (65)
$$

The spectral curves of the form (65) appear in the algebro-geometric analysis given in [10, 11].

In particular the above consideration leads to the L-A pair of the classical system different from those obtained in [9].

Theorem 9. *Under the Hess-Appel'rot conditions (5), (6), (7), the Euler-Poisson equations (1) are equivalent to the matrix equation (63), where*

$$L(\lambda) = \begin{pmatrix} 0 & -\lambda m_3 & \lambda m_2 & \gamma_1 \\ \lambda m_3 & 0 & -\lambda m_1 & \gamma_2 \\ -\lambda m_2 & \lambda m_1 & 0 & \gamma_3 + \lambda^2 \mathfrak{G}\mathfrak{M}\frac{\rho}{a_2} \\ -\gamma_1 & -\gamma_2 & -\gamma_3 - \lambda^2 \mathfrak{G}\mathfrak{M}\frac{\rho}{a_2} & 0 \end{pmatrix},$$

$$A(\lambda) = \begin{pmatrix} 0 & -\omega_3 & \omega_2 & 0 \\ \omega_3 & 0 & -\omega_1 & 0 \\ -\omega_2 & \omega_1 & 0 & \lambda\rho\mathfrak{G}\mathfrak{M} \\ 0 & 0 & -\lambda\rho\mathfrak{G}\mathfrak{M} & 0 \end{pmatrix}.$$

Remark 6. Recall that the spectral curve for the classical system given in [9] is reducible and consists of the rational and the elliptic curve. In our case, due to the involution $\tau : (\lambda, \mu) \mapsto (\lambda, -\mu)$, the singular spectral curve (65) is a two-covering of the regular (for the generic values of the integrals (2)), genus-3 hyperelliptic curve

$$\Gamma' = \Gamma/\tau : \, u^2 + uP(\lambda) + Q(\lambda)^2 = 0$$

(e.g., see Lemma 1 in [10]). In the affine part, the point $(0, 0) \in \Gamma$ is an ordinary double point as well as the ramification point of the covering.

5. Hess-Appel'rot system on $(so(4) \times so(4))^*$

Dragović and Gajić defined a class of systems of the Hess-Appel'rot type by using the analytical and algebraic properties of the classical system [11]. Subtle difference between Dragović-Gajić systems and those constructed here has to be studied carefully, but there is a big coherence between these two approaches. Remarkably, the $so(4) \times so(4)$ system given in [11] considered on the whole phase space $T^*SO(4)$ satisfies geometrical Hess-Appel'rot conditions as well.

We start from the example of the geodesic flows given in Section 3. Suppose, in addition, the G_a-invariant potential force $v(g)$ is given. Then (29) remains to be an invariant manifold for natural mechanical systems with kinetic energies $\kappa_{a,b,\mathrm{C}}$ and κ_δ. After Lagrange-Routh and partial Lagrange-Routh reductions, the systems project to the system with kinetic energy $K_{a,b}$ and potential force $V(x)$ on the adjoint orbit $\mathcal{O}(a)$, where $V(x) = v(g)$, $x = \mathrm{Ad}_g\, a$.

The reduced system is a modification of (34), (35) by the potential force:

$$\dot{x} = -\operatorname{ad}_x \operatorname{ad}_{b_x} p = [[b_x, p], x], \tag{66}$$

$$\dot{p} = -\operatorname{ad}_x^{-1}[p, [x, [b_x, p]]] - \frac{\partial V(x)}{\partial x} + \sum_{i=1}^{n} \lambda_i \xi_i(x). \tag{67}$$

Here $\xi_1(x), \dots, \xi_n(x)$ is a base of $\mathfrak{g}_x$ and Lagrange multipliers λ_i are chosen such that trajectory $(x(t), p(t))$ belongs to (32). For a proportional to b and a linear potential $V(x) = \langle x, c \rangle$, the system represents analog of spherical pendulum on adjoint orbit $O(a)$, which is integrable on an *arbitrary* orbit $O(a)$ (see Bolsinov and Jovanović [7]).

Now consider the above construction for the case $G = SO(4)$, i.e, for the 4-dimensional rigid body motion about a fixed point (we use notion from the previous section). Let

$$b = (J_1 + J_3)a, \quad a = a_{12} f_1 \wedge f_2 + a_{34} f_3 \wedge f_4. \tag{68}$$

We have $so(4)_a = \mathrm{span}\{f_1 \wedge f_2, f_3 \wedge f_4\} = so(2) \oplus so(2)$. Define

$$\mathrm{C} : so(4)_a \to so(4)_a, \quad \mathrm{C}(f_1 \wedge f_2) = 2J_1 f_1 \wedge f_2, \quad \mathrm{C}(f_3 \wedge f_4) = 2J_3 f_3 \wedge f_4.$$

Then the left-invariant kinetic energy $h_{a,b,\mathrm{C}}$ (see (27)) takes the form

$$h_{a,b,\mathrm{C}}(g,m) = J_1 m_{12}^2 + J_3 m_{34}^2 + \frac{J_1 + J_3}{2}(m_{13}^2 + m_{14}^2 + m_{23}^2 + m_{24}^2).$$

Consider the perturbation of the metric

$$h_\delta = h_{a,b,\mathrm{C}} + \delta = \frac{1}{2}\langle \mathrm{A}_\delta m, m\rangle, \tag{69}$$
$$\delta(g,m) = m_{12}(J_{14}m_{14} - J_{13}m_{23}) + m_{34}(J_{13}m_{14} - J_{14}m_{23})$$

and the potential function

$$\begin{aligned} v &= \langle \mathrm{Ad}_{g^{-1}}(a), a\rangle = \langle a_{12}e_1 \wedge e_2 + a_{34}e_3 \wedge e_4, a\rangle \\ &= \langle a, \mathrm{Ad}_g(a)\rangle = \langle a, a_{12}F_1 \wedge F_2 + a_{34}F_3 \wedge F_4\rangle \end{aligned} \tag{70}$$

(written in the coordinates (e_1, e_2, e_3, e_4) and (F_1, F_2, F_3, F_4), respectively).

The rigid body system with Hamiltonian $h = h_\delta + v$, in the left-trivialization (39), takes the form

$$\dot m = [m, \mathrm{A}_\delta(m)] + [a_{12}e_1 \wedge e_2 + a_{34}e_3 \wedge e_4, a], \quad \dot g = g \cdot \mathrm{A}_\delta(m). \tag{71}$$

The system (71) has two invariant relations

$$m_{12} = 0, \quad m_{34} = 0 \tag{72}$$

and it is reducible to the *oriented Grassmannian variety* $Gr^+(4,2) \approx O(a)$ of oriented two-dimensional planes in the four-dimensional vector space. The diffeomorphism $\Psi : O(a) \to Gr^+(4,2)$ is simply given by

$$\Psi(x) = F_1 \wedge F_2, \quad \text{where} \quad x = \mathrm{Ad}_g(a) = a_{12}F_1 \wedge F_2 + a_{34}F_3 \wedge F_4.$$

Recall that F_1, F_2, F_3, F_4 is the moving base regarded in the space frame, so $F_1 \wedge F_2$ is the oriented two-plane attached to the body.

From (33), (68), (70) we get the reduced Hamiltonian

$$H(x,p) = H_{a,b}(x,p) + V(x) = \frac{J_1 + J_3}{2}\langle [x,p], [x,p]\rangle + \langle x, a\rangle\,.$$

After calculating the Lagrange multipliers in (66), (67) the partially reduced system on $T^*Gr^+(4,2)$ become

$$\dot x = (J_1 + J_3)[[x,p],x], \tag{73}$$
$$\dot p = (J_1 + J_3)[[x,p],p] - a + \mathrm{pr}_{\mathfrak{g}_x} a. \tag{74}$$

It follows from Theorem 4 [7] that the reduced system is completely integrable by means of integrals that can be extended to commuting $SO(4)_a$-invariant functions on $T^*SO(4)$. Hence we get the following qualitative behavior of the system

Theorem 10. *The rigid body system (71) satisfies geometrical Hess-Appel'rot conditions: the partial reduction of the system from (72) to the oriented Grassmannian variety $Gr^+(4,2)$ is completely integrable pendulum type system (73), (74). Further, 10-dimensional invariant manifold (72) is almost everywhere foliated by invariant 6-dimensional Lagrangian tori that project to the 4-dimensional Liouville tori of the reduced system (73), (74). We need two additional quadratures to solve the reconstruction problem.*

By introducing $\gamma = a_{12}e_1 \wedge e_2 + a_{34}e_3 \wedge e_4$, from (71) we can write the closed system in variables (m, γ):

$$\dot{m} = [m, \omega] + [\gamma, a], \quad \dot{\gamma} = [\gamma, \omega], \quad \omega = \mathrm{A}_\delta(m). \tag{75}$$

Regarding γ as the free $so(4)$-variable, the system (75) becomes the Hamiltonian system with respect to the Lie-Poisson bracket on the dual space of the semi-direct product $so(4) \times so(4)$ (see Ratiu [25]). Moreover, with the above choice of h_δ we have

$$\mathrm{A}_\delta(m) = Jm + mJ, \quad J = \begin{pmatrix} J_1 & 0 & J_{13} & 0 \\ 0 & J_1 & 0 & J_{24} \\ J_{13} & 0 & J_3 & 0 \\ 0 & J_{24} & 0 & J_3 \end{pmatrix},$$

and (75) coincides with the Hess-Appel'rot system defined in [11].

Remark 7. A similar statement can be proved for the Hess-Appel'rot systems on $so(n) \times so(n)$ for $n > 4$ (see [11]), where one should take

$$b = (J_1 + J_3)a, \quad a = a_{12}f_1 \wedge f_2.$$

Then $so(n)_a = so(2) \oplus so(n-2)$ and the systems are partially reducible to the orbits $O(a)$, which are now diffeomorphic to the oriented Grassmannian varieties $Gr^+(n, 2)$ of oriented two-dimensional planes in the n-dimensional vector space.

Relations (72) define 6-dimensional invariant manifolds within generic symplectic leafs of $(so(4) \times so(4))^*$. Appart of the Hamiltonian function, the system (75) has a supplementary integral (see [11])

$$\mathcal{F} = \gamma_{34}a_{12} + \gamma_{12}a_{34} + (J_1 + J_3)(m_{12}m_{34} + m_{23}m_{14} - m_{13}m_{24})$$

implying the foliation on 4-dimensional invariant varieties. The L-A pair representation as well as the algebro-geometric integration of (75) up to two quadratures are given in [11].

Acknowledgments. I am very grateful to Vladimir Dragović and Borislav Gajić for useful discussions and to the referees for suggestions which help me to improve the exposition of the paper. The research was supported by the Serbian Ministry of Science, Project "Geometry and Topology of Manifolds and Integrable Dynamical Systems".

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